DYNAMICAL DECOMPOSITION OF MULTIFRACTAL TIME SERIES AS FRACTAL EVOLUTION AND LONG-TERM CYCLES: APPLICATIONS TO FOREIGN CURRENCY EXCHANGE MARKET

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The application of the multifractal formalism to the study of some time series with scale invariant evolution has given rise to a rich framework of models and processing tools for the analysis of these signals. The formalism has been successfully exploited in different ways and with different goals: to obtain the effective variables governing the evolution of the series, to predict its future evolution, to estimate in which regime the series are, etc. In this paper, we discuss on the capabilities of a new, powerful processing tool, namely the computation of dynamical sources. With the aid of the source field, we will separate the fast, chaotic dynamics defined by the multifractal structure from a new, so-far unknown slow dynamics which concerns long cycles in the series. We discuss the results on the perspective of detection of sharp dynamic changes and forecasting.

1. Introduction

The study of the statistical properties of economic time series is the focus of intense research in the last years [1,2]. There is a continuous development of new methodological tools for the characterization and processing of econometric variables. This process is very important because it allows to create models to mimic the evolution of real systems and to identify the relevant mechanisms which govern their dynamics.

Many time series of variables with econometric interest (as for instance stock market price series, option rates or foreign currency exchange rates) are known to exhibit multifractal properties [3,4,5,6,7,8]. Apart from the typical statistical properties of scale and translational invariance the main feature of multifractal systems is that they can be decomposed into a collection of sets (the fractal components) arranged in a hierarchical manner. Under changes in scale a multifractal behaves in a non-trivial way, due to the combination of scalings of each fractal component, and leading to the emergence of a collective property known as multiscaling. Almost since the origins of the multifractal formalism, two basic different approaches have been established to deal with multifractal systems. One option is to carry out an statistical analysis on the symmetries of the system, the other is to investigate the geometric aspects derived from the decomposition of the system in fractal sets.

Historically speaking the first approach is the most frequently used. It has allowed to develop a rich theory about multifractals, relating them with other properties, as energy injection cascades, intermittency and anomalous scaling (see [9] for a discussion). Nevertheless it has some difficulties. Statistical approaches are very demanding in data (multiscaling distributions are highly curtotic) and, in addition, there is no intuition about the properties of series at any specific point. Some of these problems can be solved by using a geometric approach. Methods of this type are more technical since they require an efficient decomposition of the series in the different fractal components but they are more powerful and have many advantages, as we will discuss throughout the paper. Here, we will apply the geometrical approach to the analysis of foreign currency market data series. In particular, emphasis is put on the most singular fractal component (from which the original series can be fully reconstructed) as well as on the concept of source, that allows to identify different dynamical regimes with different physical (economic) meaning.

2. Reconstructible multifractals

Let us start by giving a short theoretical introduction. To analyze a given temporal series denoted by $s(t)$ one defines a measure μ of any set A, as

$$\mu(A) \equiv \int_A d\tau \, |s'|(\tau) \tag{1}$$

where $s'(t)$ stands for the derivative of $s(t)$. The quantity $\mu(A)$ is a measure of the variations in the series $s(t)$ across the set A. The measure μ allows to locally characterize the scale invariance of the fractal components. We will say that a time series $s(t)$ is multifractal if for any Δt and any time instant t_0

$$\mu\left([t_0 - \Delta t/2, t_0 + \Delta t/2]\right) = \alpha_0 \Delta t^{d+h_0} + O(\Delta t^{d+h_0}) \tag{2}$$

where h_0 is a local exponent depending on the point t_0 considered and the shift d in the scaling has been introduced by normalization convenience; it corresponds to the dimension of the space of variables ($d = 1$ in our case)[10,11]. When a measure μ is multifractal, a scaling exponent h_0 can be assigned to every point t_0 which defines how changes of scale are locally performed. These local scaling exponents are referred to as *singularity exponents* because they characterize the functional character -*i.e.* smooth, continuous, singular...- of $s(t)$ at each one of its points. Any technique assigning a singularity exponent to every point in the series is called *singularity analysis*. The obtention of the singularity exponents at each point of an experimental, discretized series require of an appropriate interpolation scheme and of some filtering capabilities, all that usually attained by the use of wavelet projections[12,13,11,10].

Once singularity exponents are known at every time t (which we will represent by the function $h(t)$), it is possible to classify points according to common values of the exponents. We define the fractal component F_h associated to the singularity exponent h as the set of points verifying $F_h \equiv \{h(t) = h\}$; thus, those points for which local changes in scale are performed in the same way belong to the same fractal component. Once F_h are obtained, their fractal dimensions $D(h)$ can be calculated. This function, known as the *singularity spectrum* of the multifractal, plays a central role in the connection of geometrical and statistical properties of multifractals: the famous Parisi-Frisch's formula[14] express the multiscaling exponents obtained in the statistical approach in terms of $D(h)$. Hence, series can be classified according to $D(h)$.

The singularity spectrum of a physical multifractal signal is always bounded from below, that is, there always exists a finite minimum singularity exponent h_∞ since physical signals cannot display divergences to infinity. The existence of a lower bound for the singularity exponents have deep consequences for natural multifractals, as the related fractal manifold is very informative about the signal. We will call the fractal manifold associated to the most singular (*i.e.*, smallest) exponent h_∞ the Most Singular Manifold (MSM) of the multifractal hierarchy, and we will denote it by F_∞ (*i.e.*, $F_\infty \equiv F_{h_\infty}$). The MSM corresponds to rare events in the distribution of singularities, located at the tail of that distribution. But in spite of being unlike events, the dynamics of the series is lead by the MSM[10]. In fact, based on the statistical interpretation of the MSM[16] it has been proposed that multifractal signals with lower bounded spectra can be reconstructed from the values of the gradient of the signal over the MSM. An explicit reconstruction formula was derived[16], that for the case of 1D series takes the following form[10]:

$$s(t) \;=\; s(0) \;+\; \int_0^t d\tau \; s'(\tau) \tag{3}$$

Equation (3) must be understood as an integral with respect to the Hausdorff measure associated to the MSM (we assume that the MSM is a regular fractal[15]) of the values of the gradient of s restricted to the MSM. Although when this expression is applied to real, discretized signal it gives a stepwise approximation to the data, in theory more complex behaviours could be obtained from it. One of the consequences of the existence of MSM and its relevance in terms of reconstruction is that multifractal series are structured objects, containing points which convey more information than others. In addition, the series can be described in a more simplified way since the whole set of events can be coded in terms of the MSM, very useful for compression purposes [16].

3. Data. Performance of the reconstruction algorithm

For this work, we have made use of several series obtained from the foreign currency exchange market for some important currencies, sampled at a daily rate starting in January 1st, 1992 until December 31st, 1999. All the illustrations in this paper have been done for the Euro-US Dollar exchange rate. We consider the logarithm of the exchange rate, because in such a way the gradients approximate the relative variation in the exchange rate, what is the econometrically relevant quantity.

As a first step, we computed the singularity exponents associated to each time instant in the series by means of wavelet projections; see the details in[10]. When defining the fractal components on real data, we should accept an experimental quantization in the singularity determination Δh, which express the experimental uncertainty in the value. Previous works[11,10] show that quantizations of order $\Delta h = 0.1$ are acceptable for multifractal decomposition. For reconstruction task, a more exigent choice is necessary in order to attain reasonable reconstructions qualities. For the experiences in this paper, we fix a rather coarse determination of the MSM comprising the 30% of the singular points; such a amount guarantees good reconstruction qualities and efficient analysis of the series structure.

4. Reduced series

The MSM can be employed to generate a new type of signal, which is only concerned with the presence of the MSM: the reduced signal. The reduced signal $r(t)$ is generated when the reconstruction formula is applied to a simplified version of the derivatives: only the sign is kept. Hence, we take $r'(t)_{|F_\infty} = \mathrm{sign}(s'(t))$ and we assume that r is a multifractal reconstructible signal, so we apply Eq. (3) to retrieve $r(t)$. Once the reduced series is constructed, we can observe that there is a striking similarity between the original and the reduced series. Compare Figs. 1 and 2, left: a visual inspection of the graphs reveal that the reduced and the original series have locally the same shape, just different on the size or amplitude of the observed structures. This correspondence reaches further: the equivalence of multifractal structures for both series can be theoretically granted, assuming that the values of $|s'|$ change smoothly over the MSM[17]. Nevertheless, there are points where sudden differences are observed, and for which multifractal structures do not coincide. To understand the nature of these transitions points and to characterize them in a systematic way we need to introduce the concept of *source*.

5. Sources

Sources are introduced in order to provide a consistent theoretical background in support of the hypothesized continuous function which seems to relate the series s

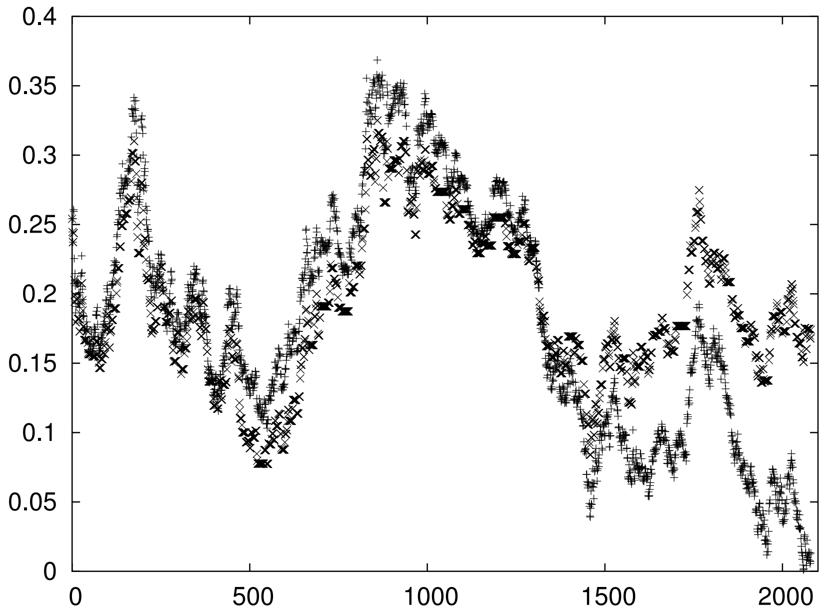

Figure 1. Original series of the logarithms of EUR-USD currency exchange rates (+) and its reconstruction from the MSM at 30% (x). The series is reconstructed in accumulated error, Eq. 3, and for that reason error increases with time.

and its reduced counterpart r. We look for a function $\rho(t)$ (that we will call the source function) which is continuous and relates the derivatives of the original series s and its reduced counterpart r, in the way:

$$s'(t) \; = \; \rho(t) r'(t) \tag{4}$$

Equation (4) has a crucial importance, as it gives an explicit representation of the original series as a combination of sources and reduced series. We can integrate both sides of the equation to obtain a new reconstruction formula, in the way:

$$s(t) \; = \; s(0) \; + \; \int_0^t d\tau \; \rho(\tau) \, r'(\tau) \tag{5}$$

This equation is in fact a generalization of the MSM-based reconstruction formula, Eq. (3), in which we have made explicit the separation between the continuous and the multifractal parts. The question is now how to obtain the source function $\rho(t)$. If we assume that both the signal $s(t)$ and the reduced series $r(t)$ define multifractal measures (denoted μ_s and μ_r) with the same singularity exponents, Eq. (5) can be interpreted in terms of Riesz's representation theorem[18], in which the measure μ_s is represented in terms of μ_r: $\mu_s(A) = \int_A d\mu_r(\tau) \; \rho(\tau) \; \forall A$. We can evaluate the integral density $\rho(t)$ as the Radon-Nikodym derivative of μ_s with respect to μ_r. Notice that if μ_s and μ_r have the same singularity exponents then $\mu_s([t-\Delta t/2, t+\Delta t/2]) \sim \Delta t^{d+h(t)} \sim \mu_r([t-\Delta t/2, t+\Delta t/2])$ and so both measures are absolutely continuous respect to each other [18]. The density $\rho(t)$ can hence be obtained by taking sequences of intervals of diminishing sizes,

$$\rho(t) \ = \ \lim_{\Delta t \to 0} \frac{\mu_s([t-\Delta t/2, t+\Delta t/2])}{\mu_r([t-\Delta t/2, t+\Delta t/2])} \tag{6}$$

The density $\rho(t)$ is called the Radon-Nikodym derivative of μ_s with respect to μ_r (sometimes represented by the symbol $\frac{d\mu_s}{d\mu_r}$), and we can identify it with the source field: Equation (6) yields Eq. (4) when the derivatives of s and r are continuous. Unfortunately, a direct application of Eq. (6) over experimental data is not easy and demands special care[17]. In a previous study over stock market data [17] the best method to evaluate the source field leads to hypothetize that the sources form a stepwise function. In this way, one relaxes our initial assumption of continuity on ρ to piecewise continuity. The piecewise continuous model we propose here is the simplest one, namely piecewise linearity: for each *source domain* A we have

$$s(t) = a_{_A} r(t) + b_{_A} \quad \forall t \in A \tag{7}$$

what implies

$$\rho(t) \ = \ a_{_A} \quad \forall t \in A \tag{8}$$

so that sources are stepwise (constant over each region; see[17] for a description on the iterative method to retrieve the source domains). There are many advantages respect to other strategies. It gives access to more regular source function ρ by definition (*i.e.*, they are piecewise constant). It gives more importance to regions than to points. The stepwise method is stable with respect to discretization. When the MSM is calculated at different densities (that is, with different quantizations), the associated stepwise sources have a very similar aspect, i.e., stepwise method is stable with respect to the quality in the determination of the MSM. Moreover piecewise constant sources grant the correspondence of multifractal structures of s and r: indeed, $\mu_s = a_{_A}\mu_r$ over A, so the singularity exponents coincide. In Figure 2, right, we show the sources calculated for our example series. Finally, the quality of the reconstructions obtained by combining the reduced series and stepwise sources is very good, even using a small number of source domains $\{A_i\}_{i=1,\dots,N}$. The results are shown in Figure 3.

When the sources are described in terms of piecewise continuous functions the series introduces some transition points (*i.e.*, the limits of the source domains) in which sharp changes take place. For the rest of points, data series are just a rescaled version of their reduced counterparts. Henceforth, the frontiers of source

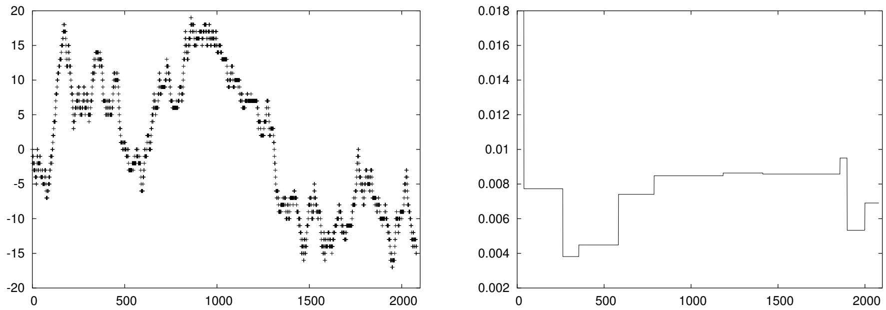

Figure 2. **Left:** Reduced series ; **Right:** Stepwise sources for EUR-USD daily series (with $N = 10$ temporal domains)

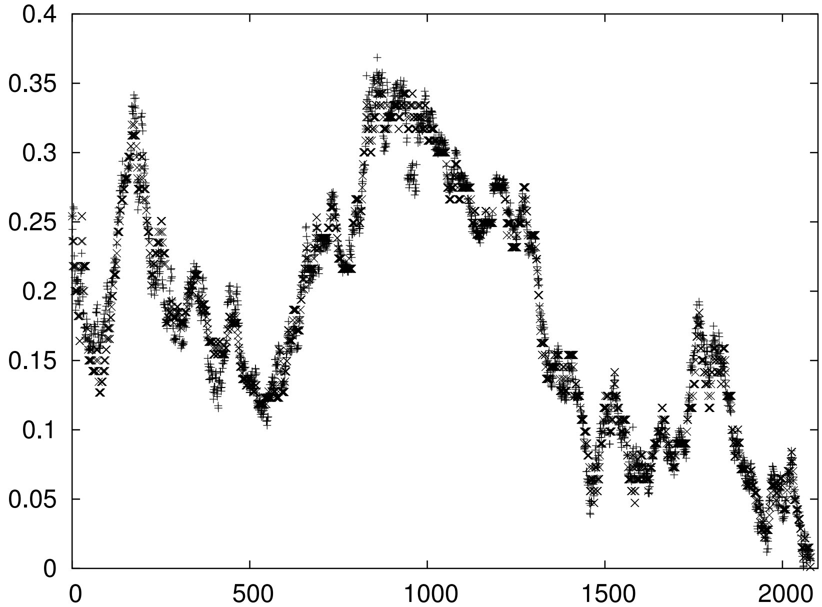

Figure 3. Original series of the logarithms of EUR-USD currency exchange rates (+) and its reconstruction applying sources (x); PSNR: 28.69 dB

domains have a dynamical interpretation: they are the transition points at which the economical system changes its behaviour. Those points are probably outside of the multifractal paradigm, as we can identify the multifractal structures of the original and the reduced series except for those points. Transition points in sources can probably be interpreted as too singular points in which statistics of temporal series deviates from multifractality, so they should be studied separately. Transition points are very relevant in order to understand changes in the dynamics of the system; for instance, in other contexts as for instance meteorology the presence of transition source points have been related with the presence of convection currents, precipitations and other effects related with strong energy exchanges [19]. In our context, source plateaus are very close in value and transitions have small amplitudes, in contrast to what is observed in the study of stock market series [17]. A possible interpretation of source values and its transitions is the following. Let us consider the constant value a_A at each source domain A. By definition (see Eq. (4)) the absolute values of the derivatives of the series over the MSM equal that constant. The constant a_A represents then the maximum expected change per unit of time during the whole domain A. In this sense, a_A gives a measure of the maximum volatility[a] over that period: the volatility $\sigma(t_0, \Delta t)$ at a given time t_0 and over a window length Δt will be given by:

$$\sigma(t_0, \Delta t) \; = \; a_A \, \sigma_r(t_0, \Delta t) \tag{9}$$

where $\sigma_r(t_0, \Delta t)$ is the volatility for the reduced signal, that we will call reduced volatility. Taking into account that at any time step the reduced series can change at most by 1, $\sigma_r(t_0, \Delta t) < \Delta t$. According to our model, it follows that at each source domain $\sigma(t_0, \Delta t)/\Delta t < a_A$, which is consistent with the interpretation of a_A as maximum volatility per unit of time given above.

In spite of the simplicity of the reduced signal, in our model the volatility is a complicated, random variable because the reduced volatility itself is random. The reduced volatility depends on local variations in tendency (which depend on the particular basis point t_0 as well as on the size of the window extent Δt), on the Markov parameters characterizing the reduced series [10] (which in turn depend on the particular source domain A containing t_0) and in the multifractal structure (which does not evolve but may depend on the particular series we are studying; however this later dependency is integrated in the Markov parameters and does not require a separate study). An interesting prediction of our model is that all cycles and variations of volatility are also cycles and variations of reduced volatility, as both volatilities just differ in piecewise constant scale factors. In Figure 4 we observe that such prediction is correct over our example series. Notice that the very similar values in the source plateaus observed for the Euro-US Dollar exchange rate would imply that this market has a rather constant distribution of volatilities.

6. Conclusions and future work

The experimental evidence confirms that relevant econometric variables, such as volatility, can be characterized over long periods (the source domains) by the reduced series, which just contains geometrical information. The results confirm the

[a]Here we define the volatility at a given point as the standard deviation of values of series for a window of fixed size around that point.

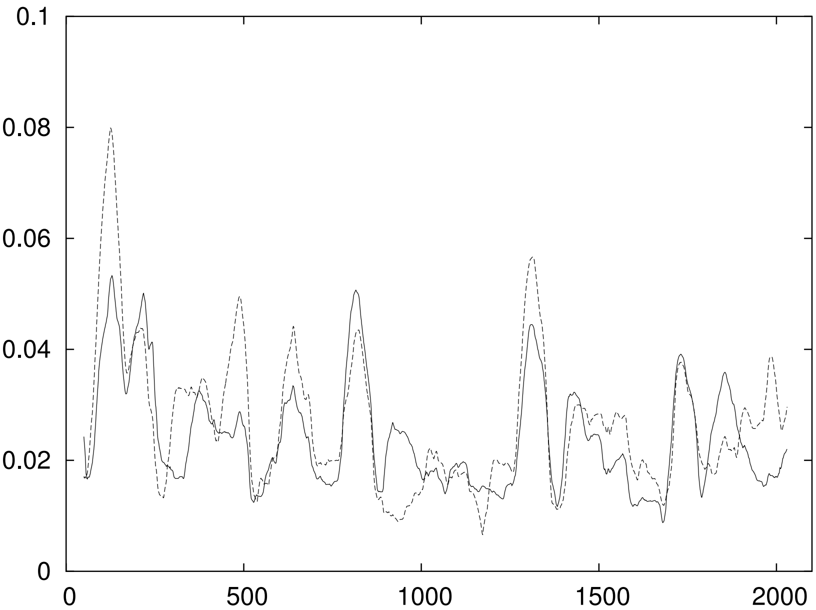

Figure 4. Volatility time series for the original data (continuous line) and the reduced counterpart (re-scaled by a factor 0.01 to fit the graph; dashed); $\Delta t = 100$ ticks. Notice that we can identify the same cycles for both volatility series, in agreement with theory.

importance of the MSM, as this set characterizes completely the reduced series. Any model for the volatility should primarily consist of a model for the reduced series (as the one presented in [10]). The sources have an important role, also. According to our model, volatilities must be recalculated once a transition point is crossed: following Eq. (9), the constant relating the reduced volatility with the actual volatility changes after each transition - as it is the source term. This fact makes the detection of transition points crucial.

The results presented here are obtained on the basis of a *descriptive* analysis, that is, the structure and the properties we have isolated are obtained by analyzing the series *a posteriori*: when processing a time instant in the series, we take into account both the past and the future times. But there is nothing in the multifractal formalism which prevents the use of unilateral intervals to analyze local scalings as in Eq. (2): namely:

$$\mu([to - \Delta t/2, t_0]) \; = \; \alpha_0 \Delta t^{d+h_0} \; + \; O(\Delta t^{d+h_0}) \tag{10}$$

Preliminary results on this type of *prospective* analysis (*i.e.*, oriented to the prediction) shows that the determination of the multifractal structure is still possible, although the experimental uncertainties increase, specially in the detection of points in the MSM. Timely detection of the MSM have relatively important uncertainties, which decay fast as time passes and the point under study is further in the past of the last accessible point.

Concerning prospective detection of source domains, a new algorithm needs to be define, as the one presented[17] is by construction strictly descriptive and the determination of source domains is made in a global basis and not locally. This is an important limitation of the algorithm which is not intrinsic to sources; in fact, sources are local, as they correspond to the Radon-Nikodym derivative. New stable, unilateral methods for the obtention of sources on experimental, discretized data need to be implemented.

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